Appendix E

Karmarkar's Method

E.1 Introduction

In 1984 introduced a new and innovative polynomial-time algorithm for linear programming. The polynomial running-time of this algorithm combined with its promising performance created tremendous excitement (as well as some initial skepticism) and spawned a flurry of research activity in interior-point methods for linear programming that eventually transformed the entire field of optimization.

Despite its momentous impact on the field, Karmarkar's method has been superseded by algorithms that have better computational complexity and better practical performance. Chapter 10 presents an overview of some of the leading interior point methods for linear programming. Karmarkar's method still remains interesting because if its historical impact, and possibly, because of its projective scaling approach. This Appendix outlines the main concepts of the method.

E.2 Karmarkar's Projective Scaling Method

In Section 10.5 we introduced the primal affine-scaling method for solving linear programs. The basic idea of an iteration is to transform the linear problem via an affine scaling, so that the current point x_k is transformed to the "central point" $e = (1, 1, \ldots, 1)^T$, then to take a step along the steepest-descent direction in the transformed space, and finally to map the resulting point back to its corresponding position in the original space. Conceptually, Karmarkar's algorithm may be described in similar terms, except that a "projective scaling" transformation is used. Some special assumptions must be made, however.

The first, which is standard, is that the linear program has a strictly feasible point, and that the set of optimal points is bounded. The second assumption is that the linear program has a special "canonical" form:

$$\begin{array}{ll} \text{minimize} & z = c^T x \\ \text{subject to} & Ax = 0 \\ & a^T x = 1, \\ & x \geq 0. \end{array}$$

This assumption is not restrictive, and any linear program can be written in such form. Consider for example a problem in standard form

$$\begin{array}{ll} \text{minimize} & \tilde{c}^T \tilde{x} \\ \text{subject to} & \tilde{A}\tilde{x} = \tilde{b}, \\ & \tilde{x} \geq 0 \end{array}$$

and assume for convenience that $\tilde{x}$ has dimension $n-1$. By introducing a new variable x_n that is always equal to 1, we can write the problem as

$$\begin{array}{ll} \text{minimize} & z = \tilde{c}^T \tilde{x} \\ \text{subject to} & \tilde{A}\tilde{x} - \tilde{b}x_n = 0, \\ & x_n = 1, \\ & \tilde{x}, x_n \geq 0. \end{array}$$

The problem is now in the special canonical form, with $A = [\tilde{A}, -\tilde{b}]$, $x = (\tilde{x}, x_n)^T$, $a = e_n$ (the n-th unit vector), and $c = (\tilde{c}, 0)^T$.

The third assumption is that the value of the objective at the optimum is known, and is equal to zero. This is an unlikely assumption that is not likely to hold. Although it appears restrictive, it is possible, nevertheless, to adapt the method to solve problems where the optimal objective value is unknown. We address this issue later in the Section.

An example of a program that satisfies the three assumptions is

$$\begin{array}{ll} \text{minimize} & z = x_1 - 3x_2 + 3x_3 \\ \text{subject to} & x_1 - 3x_2 + 2x_3 = 0 \\ & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

The first two assumptions are clearly satisfied. The optimal solution is $x_* = (3/4, 1/4, 0)^T$ with corresponding objective $z_* = 0$, and so the third assumption is satisfied also.

Karmarkar's algorithm starts at an interior feasible point. At each iteration of the algorithm: (i) the problem is transformed via a *projective transformation*, to obtain an equivalent problem in transformed space, (ii) a projected steepest-descent direction is computed, (iii) a step is taken along this direction, and (iv) the resulting point is mapped back to the original space. We discuss each of these steps in turn.

The projective transformation in Karmarkar's method can be split into two operations: the first scales the variables as in the affine scaling, so that the current point x_k goes to e; the second scales each resulting point by the sum of its variables. The end result is that x_k is transformed to $e/n = (1/n, \ldots, 1/n)^T$, and the sum of the components at any point in the transformed space is equal to 1. Mathematically the projective transformation sends x to

$$\bar{x} = \frac{X^{-1}x}{e^T X^{-1} x},$$

where $X = \operatorname{diag}(x_k)$. The corresponding inverse transformation is

$$x = \frac{X\bar{x}}{a^T X\bar{x}}.$$

Example E.1 Projective Transformation. Consider the constraints

$$\begin{aligned} &x_1 - 3x_2 + 2x_3 = 0 \\ &x_1 + \ \ x_2 + \ \ x_3 = 1 \\ &x_1, x_2, x_3 \ge 0. \end{aligned}$$

The feasible region is the line segment between $x_a = (3/4, 1/4, 0)^T$ and $x_b = (0, 2/5, 3/5)^T$. Notice that $e/n = e/3 = (1/3, 1/3, 1/3)^T$ is indeed feasible. Let $x_k = (1/8, 3/8, 1/2)^T$. The projective transformation that takes x_k to $e/3$ can be performed in two steps. First the affine transformation takes x to

$$X^{-1}x = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8/3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8x_1 \\ 8/3x_2 \\ 2x_3 \end{pmatrix}$$

and next this point is scaled by the sum of its variables. The final image of x under this transformation is

$$\bar{x} = \frac{1}{8x_1 + 8/3x_2 + 2x_3} \begin{pmatrix} 8x_1 \\ 8/3x_2 \\ 2x_3 \end{pmatrix}.$$

Thus x_a is sent to $\bar{x}_a = (\frac{9}{10}, \frac{1}{10}, 0)^T$, x_b is sent to $\bar{x}_b = (0, \frac{8}{11}, \frac{3}{11})^T$, and $e/3$ is sent to $(\frac{12}{19}, \frac{4}{19}, \frac{3}{19})^T$, which lies on the line segment between $\bar{x}_a$ and $\bar{x}_b$.
■

The projective transformation transforms the original problem into

$$\begin{aligned} \underset{\bar{x}}{\text{minimize}} \quad & \frac{c^T X\bar{x}}{a^T X\bar{x}} \\ \text{subject to} \quad & AX\bar{x} = 0 \\ & e^T\bar{x} = 1, \\ & \bar{x} \ge 0. \end{aligned}$$

(See the Exercises.) The new objective function is not a linear function. However, since the optimal objective value in the original problem is assumed to be zero, the optimal objective in the new problem will also be zero. The denominator is positive and bounded, so the optimal value of the numerator will be zero as well. As a result, we will ignore the denominator, and consider the problem

$$\begin{array}{ll} \text{minimize} & \bar{c}^T\bar{x} \\ \text{subject to} & \bar{A}\bar{x} = 0 \\ & e^T\bar{x} = 1, \\ & \bar{x} \ge 0, \end{array} \qquad (\bar{P})$$

where $\bar{c} = Xc$ and $\bar{A} = AX$.

We now compute the projected steepest-descent direction for problem $\bar{P}$. Denoting the constraint matrix by

$$B = \begin{pmatrix} \bar{A} \\ e^T \end{pmatrix} = \begin{pmatrix} AX \\ e^T \end{pmatrix},$$

the corresponding orthogonal projection matrix is $P_B = I - B^T(BB^T)^{-1}B$. Since $(AX)e = Ax_k = 0$,

$$P_B = \bar{P} - \frac{1}{n}ee^T.$$

(See the Exercises.) The projected steepest-descent direction is $\Delta\bar{x} = -P_B\bar{c} = -P_BXc$. Using the previous equation and the relation $e^TXc = x_k^Tc = c^Tx_k$ we obtain the following expression for the direction in transformed space:

$$\Delta\bar{x} = -\bar{P}Xc + \frac{c^Tx_k}{n}e.$$

Starting from e/n, we now take a step of length α along the projected steepest-descent direction:

$$\bar{x}_{k+1} = \frac{e}{n} + \alpha\Delta\bar{x}.$$

The first requirement on α is that the new point satisfy the nonnegativity constraints. Any step length less than $\alpha_{\max}$, the step to the boundary, will fulfill this requirement. This requirement alone does not guarantee polynomial complexity. Karmarkar proposed a suitable step length, by inscribing a sphere with center e/n in the set $\{\, x : e^T\bar{x} = 1, \bar{x} \ge 0 \,\}$. He showed that the largest such inscribed sphere has radius $r = (n(n-1))^{-1/2}$. By taking a step θr along the scaled direction $\Delta\bar{x}/\|\Delta\bar{x}\|$ with $0 < \theta \le 1$, feasibility is always maintained. However, to guarantee polynomial complexity, θ must be restricted to a rather small interval, with $\theta = 1/3$ being an acceptable choice. In practice, better progress can often be made by taking a much larger step, slightly less than $\alpha_{\max}$, even though the polynomial complexity may be lost.

The final step of an iteration is to map $\bar{x}_{k+1}$ back to x-space. This gives the new solution estimate

$$x_{k+1} = \frac{X\bar{x}_{k+1}}{a^TX\bar{x}_{k+1}}.$$

Example E.2 Iteration of Karmarkar's method. Consider the previous example, and suppose as before that the current point is $x_k = (1/8, 3/8, 1/2)^T$. Then

$$AX = \begin{pmatrix} 1/8 & -9/8 & 1 \end{pmatrix} \quad \text{and} \quad \bar{c} = Xc = \begin{pmatrix} 1/8 \\ -9/8 \\ 3/2 \end{pmatrix},$$

hence

$$\bar{P}Xc = \begin{pmatrix} 0.1113 \\ 1.2483 \\ 1.3904 \end{pmatrix}, \qquad \frac{c^T x_k}{n} e = \frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

and the projected steepest-descent direction is

$$\Delta \bar{x} = -\bar{P}Xc + \frac{c^T x_k}{n} e = \begin{pmatrix} 0.0554 \\ -1.0816 \\ -1.2237 \end{pmatrix}.$$

Suppose we decide to use a stepsize of $0.9\alpha_{\max}$. Here

$$\alpha_{\max} = \min \left\{ \frac{(1/3)}{1.0816}, \frac{(1/3)}{1.2237} \right\} = \min \{ 0.3082, 0.2724 \} = 0.2724.$$

The resulting step length is $\alpha = 0.9 \times 0.2724 = 0.2451$. The new point in transformed space is

$$\bar{x}_{k+1} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} + 0.2451 \begin{pmatrix} 0.0554 \\ -1.0816 \\ -1.2237 \end{pmatrix} = \begin{pmatrix} 0.3469 \\ 0.0682 \\ 0.0333 \end{pmatrix}.$$

Transforming back to original space we obtain

$$x_{k+1} = \frac{1}{0.0856} \begin{pmatrix} 0.3469 \\ 0.0682 \\ 0.0333 \end{pmatrix} = \begin{pmatrix} 0.5066 \\ 0.2987 \\ 0.1947 \end{pmatrix}.$$

The objective value at the new point is $c^T x_{k+1} = 0.7102$. ∎

A difficulty with Karmarkar's method is the assumption that the optimal objective value must be zero. If the optimal objective value z_* were known, we could replace the objective function $c^T x$ by $(c - z_* a)^T x$, since this is equal to $c^T x - z_*$ for any feasible x, and the the optimal objective value would be zero. In practice z_* is not known. However it is possible to adapt this approach, using lower bounds on z_* obtained from dual solutions.

Given a lower bound w_k, the vector of objective coefficients is replaced by $(c - w_k a)$ and accordingly, the vector of objective coefficients in $\bar{P}$ is replaced by $\bar{c}(w_k) = X(c - w_k a)$. The dual to $\bar{P}$ is then

$$\begin{array}{ll} \text{maximize} & w \\ \text{subject to} & \bar{A}^T y + ew \le \bar{c}(w_k). \end{array}$$

If w_k were indeed equal to z_*, the optimal value for w would be zero. An estimate for the optimal vector y is

$$y_k = (\bar{A}\bar{A}^T)^{-1}\bar{A}\bar{c}(w_k).$$

The maximal value of w for which (y_k, w) is dual feasible is equal to the minimum component $\bar{w}$ of $\bar{c}(w_k) - \bar{A}^T y_k$. If $\bar{w} \geq w_k$, then the new lower bound w_{k+1} is set to $\bar{w}$; otherwise $w_{k+1} = w_k$.

To prove polynomiality of the algorithm, Karmarkar introduced a potential function

$$f(x) = n \log c^T x - \sum_{j=1}^{n} \log x_j = \sum_{j=1}^{n} \log(c^T x / x_j).$$

He proved that this function is reduced by a constant at each iteration. (The constant, which we denote by γ, depends on the stepsize parameter θ.) This implies that, after k iterations,

$$c^T x_k \leq e^{-k/\gamma n} c^T x_0.$$

Since the algorithm terminates if $c^T x_k < 2^{-2L} c^T x_0$, it is possible to show that the number of iterations is at most $O(nL)$. Although the potential function is guaranteed to decrease at each iteration, there is no such guarantee for the objective $c^T x$ and, in fact it may occasionally increase.

In a 1986 paper, Gill, Murray, Saunders, Tomlin, and Wright showed that Karmarkar's method is equivalent to the logarithmic barrier method with a particular choice of the barrier parameter (which, atypically for barrier methods, may be negative). To verify this, we obtain an explicit expression for the search direction in the original space. First,

$$\bar{x} = \frac{e}{n} - \alpha \bar{P} X c + \alpha \frac{c^T x_k}{n} e = \frac{1 + \alpha c^T x_k}{n} e - \alpha \bar{P} X c.$$

From this it follows that

$$X\bar{x} = \frac{1 + \alpha c^T x_k}{n} x_k - \alpha X \bar{P} X c.$$

The new point x_{k+1} is obtained by scaling $X\bar{x}$ by $a^T X \bar{x}$. Denoting the latter by τ,

$$\tau = a^T X \bar{x} = \frac{1 + \alpha c^T x_k}{n} - \alpha a^T X^T \bar{P} X c,$$

since $a^T x_k = 1$. Denoting $\mu_k = a^T X^T \bar{P} X c$, this gives

$$\tau = \frac{1 + \alpha c^T x_k}{n} - \alpha \mu_k.$$

We now obtain

$$\begin{aligned} x_{k+1} = \frac{X\bar{x}}{\tau} &= \frac{1 + \alpha c^T x_k}{\tau n} x_k - \frac{\alpha}{\tau} X \bar{P} X c \\ &= (1 + \frac{\alpha \mu_k}{\tau}) x_k - \frac{\alpha}{\tau} X \bar{P} X c \end{aligned}$$

$$= x_k + \frac{\alpha\mu_k}{\tau}\left(x_k - \frac{1}{\mu_k}X\bar{P}Xc\right).$$

If we define

$$\alpha_k = \frac{\alpha\mu_k}{\tau}, \quad \Delta x = x_k - \frac{1}{\mu_k}X\bar{P}Xc$$

then

$$x_{k+1} = x_k + \alpha_k \Delta x,$$

so the iteration is now written as a regular search algorithm. The last technicality is to observe that $x_k = X\bar{P}e$. Therefore we can write

$$\Delta x_k = -\frac{1}{\mu_k}X\bar{P}Xc + X\bar{P}e, \quad \text{where } \mu_k = x_k^T\bar{P}Xc.$$

This is exactly the search direction prescribed by the primal logarithmic-barrier path-following algorithm for problems in standard form (see Section 4). It is also the search direction for the logarithmic barrier function for problems that are given in the canonical form (see the Exercises).

From a practical point of view, Karmarkar's method and its variants have not been as successful as other interior-point methods. In numerical tests they appear to be slower and less robust than leading methods. A possible explanation for this less satisfactory performance is the need to generate lower bounds w on the optimal objective. These bounds can be poor, in particular if the dual feasible region has no interior, and this can cause the method to converge slowly.

Exercises

2.1. In Karmarkar's method prove that $AXe = 0$.

2.2. Let

$$B = \begin{pmatrix} AX \\ e^T \end{pmatrix}.$$

Prove that $P_B = \bar{P} - \frac{1}{n}ee^T$. Hint: Use the result of the previous Problem. Prove also that $P_B = P_{e^T}\bar{P}$.

2.3. Consider the logarithmic barrier function for a problem in Karmarkar's special form. Show that if $\mu_k = x_k^T\bar{P}Xc$, then the search direction in Karmarkar's method coincides with the projected Newton direction for the barrier function at x_k. (Hint: Prove that the Karmarkar search direction satisfies the projected Newton equations.)

E.3 Notes

In his original paper Karmarkar used $e^Tx = 1$ as the normalizing constraint in the canonical form. He showed that all linear programs can be transformed to this specific form, but the proposed transformation results in a much larger linear program. Variants of the method that are suitable for problems in standard form were proposed by Anstreicher (1986), Gay (1987), and Ye and Kojima (1987).